# Intermittency - Basic Ideas

Percolation Theory, etc. aimed at developing model of transport in intermittent systems, so, first discuss intermittency

→ Intermittent ≡ distribution concentrated in widely spaced patches where pdf >> $\overline{pdf}$.

→ patchy, bursty

→ implies departure from familiar world of CLT, LLN, Gaussian statistics, i.e. smooth distribution.

(a) Recall CLT:

Let $x_1, x_2, \ldots$ be a sequence of
- independent
- identically distributed

random variables, each with mean $\mu$, and variance $\sigma$, so:

$$\overline{x_i} = \mu$$

$$\langle (x_i - \overline{x_i})^2 \rangle = \sigma^2$$

then $\frac{\overset{\text{additive}}{\overset{\downarrow}{x_1 + x_2 + \dots x_N}} - n\mu}{\sqrt{n}\,\sigma}$ is distributed

as a $\overset{\text{sum}}{\overset{\downarrow}{\text{Gaussian}}}$;

$\Rightarrow$ sum converges to Gaussian dist of width $\sqrt{N}\,\sigma$.

Point:

- no correlations
- identically distributed i.e. no "special" steps $\left\{ \text{each } x_i \text{ has same distribution} \right.$
- $\sigma^2$ exists $\Rightarrow$ no fat tails.

$\Leftrightarrow$ additive process $\Rightarrow$ sum has Gaussian distribution.

b.)

Now, consider multiplicative process:

- define $X \equiv \prod_{i=1}^{N} x_i = x_1 x_2 \dots x_j \dots x_N$

- $x_j = \begin{cases} 0 & p = 1/2 \\ 2 & p = 1/2. \end{cases}$

→ what of $\langle x \rangle$, $\langle x^2 \rangle$ ?

**Point:** $x = 0$ unless all $x_i = 2$, then
$x = 2^N$. $P(x = 2^N) = 2^{-N}$.

so

$$\langle x \rangle = \sum_{\substack{j \\ \text{realiz.}}} x_j \Big/ \underbrace{2^N}_{\#\text{ realizations}}$$

{here $j$ denotes realization of $x$

$$= 0 + 0 \ldots + 2^N / 2^N = 1$$

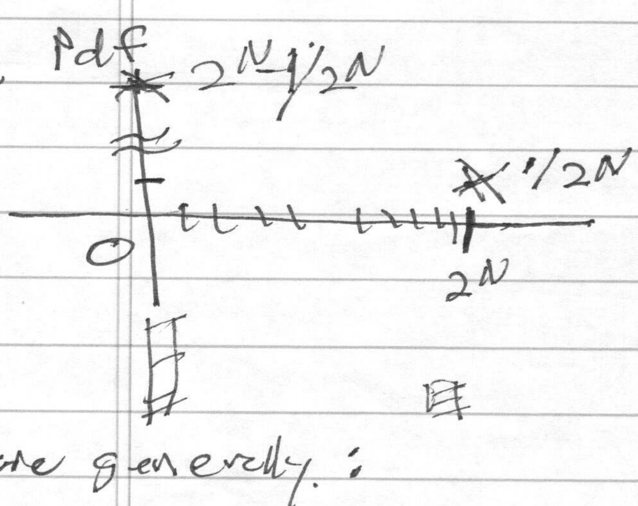
($j \leftrightarrow$ realization)

$$\langle x^2 \rangle = \sum_j x_j^2 / 2^N$$

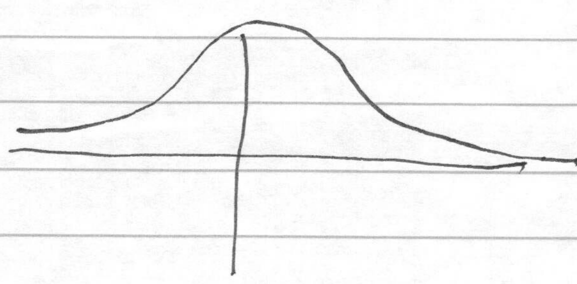
$$= 0 + 0 + 0 + \ldots 2^{2N} / 2^N = 2^N.$$

concentrated pdf, with tail

i.e.




contrast:



More generally:

$$\langle x^P \rangle \cong 2^{(P-1)N}$$

{higher moments grow

signature of intermittent process is growth higher moments

# N large

so, for growth

$$\gamma_p = \frac{\log_2 \langle X^p \rangle}{N} = p-1$$

N is surrogate for time, # steps

$\Rightarrow$ higher moments grow $\sim p$

- X is intermittent random quantity.

- intermittent random quantity is result of multiplying, not adding, many random numbers.

Now,

- $x_j$ = random # distributed around unity

- $$X = \prod_{j=1}^{N} x_j$$

multiplicative process

$$\Rightarrow \quad \ln X = \ln x_1 + \ln x_2 + \ldots \ln x_N$$

Now, apply CLT to sum of logs:

Now $x \to \mathcal{E}$ (notation change)

so, as $N \to \infty$

$$P(\ln \mathcal{E}) \sim \exp\left[-(\ln \mathcal{E})^2 / N \eta^2\right]$$

$$\ln \mathcal{E} \sim N^{1/2} \eta$$

$\eta$: Gaussian random quantity, unit dispersion

$$\Rightarrow \boxed{\mathcal{E} \sim \exp(N^{1/2} \eta)} \quad \left\{ \begin{array}{l} \text{log normal} \\ \text{distribution} \end{array} \right.$$

$\eta$: on $(-\mu\sigma, \mu\sigma)$, $\mu \sim 1$

specific realization, $\sigma \sim$ std dev. $\eta$.

so

- $\eta < 0$, $\mathcal{E}$ exponentially small
- $\eta > 0$, $\mathcal{E} \sim \exp(N^{1/2} \sigma)$, (yuge!)

$\Rightarrow$ spikey distribution

- $\langle \mathcal{E} \rangle = \int \mathcal{E}(\eta) P(\eta)$

$$= \int \exp\left[N^{1/2} \eta\right] \exp\left[-\eta^2 / 2\sigma^2\right]$$

$$\sim \exp \underbrace{N\sigma^2/2}_{\text{exponentially growing avg}} \quad \left\{ \begin{array}{l} \text{avg} \\ \text{grows} \\ \text{with } N \end{array} \right.$$

$\Rightarrow$

$$- \langle \varepsilon^p \rangle \sim \exp\left[ p^2 N \sigma^2 / 2 \right]$$

higher moments grow exponentially !

$$- \gamma_p \sim \lim_{N \to \infty} \frac{\ln \langle \varepsilon^p \rangle}{N} = \underline{\underline{p^2 \sigma^2 / 2}}.$$

rapid growth higher moments $\Rightarrow$ symptom concentration

$\Rightarrow$ grows $\sim p^2$

Log-normal : prototype [redacted] { multiplicative intermittent random var. distribution

$$- \ln \varepsilon \sim N^{1/2} \eta$$

$$\varepsilon \sim \exp\left[ N^{1/2} \eta \right]$$

$$- \langle \varepsilon \rangle \sim \exp\left[ N \sigma^2 / 2 \right]$$

c.) Evolution of Random Quantity

- Multiplicative random quantities arise in evolutionary problems. $N \leftrightarrow$ time.

- Consider simple comparison :

$$d\psi/dt = \varepsilon(t)$$

$\varepsilon(t) \equiv$ noise
Gaussian dist, disp'n $\sigma^2$
$\sim$ delta corr !

and $\Rightarrow$ clearly diffusion process.
$\rightarrow$ additive noise

$$\frac{d\Psi}{dt} = \mathcal{E}(t)\,\Psi$$

{ stochastic pde
{ $\rightarrow$ NL model

$\rightarrow$ multiplicative noise.

Now, $\dfrac{d\Psi}{dt} = \mathcal{E}(t)$

$$\langle \Psi^2 \rangle \sim \int^{t_1} \mathcal{E}(t_1)\,dt_1 \int^{t_2} \mathcal{E}(t_2)\,dt_2$$

$$\sim \overline{\mathcal{E}^2}\,\tau_c\, t \quad \checkmark$$

in detail

assume $\mathcal{E}$ re-sets every $\tau$, so:

$$\Psi = \int \mathcal{E}(t)$$

$$= \int_0^{\tau} \mathcal{E}(t)\,dt + \int_{\tau}^{2\tau} \mathcal{E}(t)\,dt + \ldots$$

CLT $\Rightarrow$

$$= \langle \mathcal{E} \rangle t + \overset{\text{norm.}}{\tau}\;\underbrace{\sigma}_{\mathcal{E}_{rms}}\;\overbrace{\left(t/\tau\right)^{1/2}}^{N^{1/2}}\;\underbrace{\eta}_{\text{Gaussian dist}}$$

$$\overline{\eta^2} \sim 1$$

ignoring $\langle \mathcal{E} \rangle$

$$\langle \Psi \rangle^2 \sim \tau^2 \sigma^2 \frac{t}{\tau} \langle \eta^2 \rangle \quad \checkmark$$

$$\sim \tau \sigma^2 t$$

Now, multiplicative case:

$$\frac{d\Psi}{dt} = \mathcal{E}(t)\,\Psi$$

{ here linear
{ stochastic pde
{ as model of
{ NL pde

$$\frac{d}{dt} \ln \Psi = \mathcal{E}(t)$$

$$\psi(t) \sim \psi_0 \exp\left[\int_0^t \varepsilon(s)\, ds\right]$$

$$\sim \prod_{n=0}^{t/\tau} \exp \int_{n\tau}^{(n+1)\tau} \varepsilon(s)\, ds \quad \Rightarrow \text{multiplicative}$$

$\ln\psi$ follow CLT:

$$\boxed{\ln\psi \sim \langle\varepsilon\rangle t + \sigma\tau (t/\tau)^{1/2} \eta}$$

$\Rightarrow \langle\varepsilon\rangle = 0$

$$\boxed{\psi \sim \exp\left[\sigma\tau (t/\tau)^{1/2} \eta\right] \Rightarrow \text{log normal}}$$

- $\psi$ grows as $\exp(t^{1/2})$
- solution is intermittent, random quantity
- fluctuations grow in time as $\exp\left[(t/\tau)^{1/2}\right]$.

- compare to closure calculation!?

Naive approach would be:

$$\frac{d\psi}{dt} = \varepsilon \psi$$

$$\psi = \langle \psi \rangle + \tilde{\psi}$$

$$\frac{d}{dt}\langle \psi \rangle = \overline{\varepsilon \tilde{\psi}}$$

where $\dfrac{d\tilde{\psi}}{dt} = \varepsilon \langle \psi \rangle$

$$\Rightarrow \frac{d\langle \psi \rangle}{dt} = \overline{\varepsilon \int \varepsilon \langle \psi \rangle}$$

for short $\tau_c$

$$\cong \overline{\varepsilon^2} \tau_c \langle \psi \rangle$$

$$\langle \psi \rangle \sim \psi_0 \exp\left[\overline{\varepsilon^2} \tau_c t\right]$$

$$\sim \psi_0 \exp\left[\nabla^2 \tau_c t\right]$$

and if take:

$$\psi \sim \exp\left[\nabla \tau (t/\tau)^{1/2} \eta\right]$$

$$\langle \psi \rangle \sim \left\langle \exp\left(\nabla \tau (t/\tau)^{1/2} \eta\right) \exp(-\eta^2) \right\rangle$$

then:

$$\langle \psi \rangle \sim \int d\eta \; e^{-\eta^2} e^{c\eta}$$

$$\sim \int d\eta \; \exp\left[-\left(\eta^2 - c\eta + \frac{c^2}{4} - \frac{c^2}{4}\right)\right]$$

$$\sim \int d\eta \; \exp\left[-(\eta - c/2)^2\right] e^{c^2/4}$$

$$\sim \# \exp\left[v^2 T^2 t/\ell^2\right] \Rightarrow \text{agrees!}$$

Point: → Mean field / QLT will get average right

but → won't reveal fundamental nature and structure of higher moments.

so

Intermittency → $\left\{ \boxed{\text{occurrence of rare but intense peaks in behavior of random quantity.}} \right.$

The Question: what are statistics of particle/density distribution in random media?

Now → Random Media [
→ continuum of random quantities.

Consider: Random potential:

spatial structure

$U(x, \omega)$ at temperature $T$

defines random media

{ enumerates realizations

{ $\omega$ fixed → definite position

$U$ → Gaussian dist.

→ $\sigma^2$

Now, ⇒ distribution

$$n = n_0 \exp\left[-U/k_B T\right]$$

density occupation

→ dist. $n$ non-Gaussian as dep. on $U$ is non-linear.

Now, for most probable concentration:

$$P(n(U)) = n(U)\, p(U)$$

$$= n_0 \exp\left[-U/k_B T\right] e^{-U^2/2\sigma^2}$$

and seek max $P$.

so $\begin{cases} \rho_{max} = n_0 \exp[\Gamma^2/2k_b^2 T] \\ \downarrow \\ \text{max probability} \\ \text{at } U_{max}/\Gamma \sim -\Gamma/k_b T \end{cases}$

and look at higher moments of density dist.

$$\langle n \rangle \equiv n_0 \exp[\Gamma^2/2k_b^2 T^2]$$

$$\langle n^2 \rangle^{1/2} = n_0 \exp[\Gamma^2/k_b^2 T^2]$$

$$\langle n^p \rangle^{1/p} = n_0 \exp[p\Gamma^2/2k_b^2 T^2]$$

note:

$$\langle n^2 \rangle \gg \langle n \rangle^2 \quad \text{i.e.} \quad \rightarrow \begin{array}{l} n_{rms} \gg \langle n \rangle \\ \text{fluctuation dominated system} \end{array}$$

$$\checkmark \; n_0^2 \exp(2\Gamma^2/k_b^2 T^2) \gg n_0^2 \exp[\Gamma^2/k_b^2 T^2]$$

$\Gamma^2/k_b^2 T^2 > 1$

$$\langle n^4 \rangle \gg \langle n^2 \rangle^2$$

$$\langle n^4 \rangle = n_0^4 \exp[16\Gamma^2/2k_b^2 T^2] \gg$$

$$n_0^4 \exp[4\Gamma^2/k_b^2 T^2] \quad \checkmark$$

$\Rightarrow \langle n^2 \rangle > \langle n \rangle^2$

etc $\Rightarrow$ higher moments larger

$\Rightarrow$ successive mean not determined by most probable $n$, $\Gamma/k_B T$

but by $\boxed{p^{1/2}\, \Gamma/k_B T}$

$\Rightarrow$ signatures of intermittent distribution of density.

Note: why the interest in higher moments?

$\Rightarrow$ "Progressive growth of statistical moments with order can be explained only ~~[illegible]~~ by a much more pronounced dominance of rare intense peaks in the concentration distribution".

$\Rightarrow$ Linear Scalar Equations:
Random Growth (Potential) + diffusion.

$\Rightarrow$ statistics

$\Rightarrow$ moments.

Stochastic reaction-diffusion:

have: $$\frac{\partial \psi}{\partial t} = \underbrace{D\nabla^2\psi}_{\text{diffusion}} + \underbrace{u(x,\omega)\,\psi}_{\text{stoch. reaction}}$$

$$\psi(x,0) = \psi_0 \qquad (\text{I.C.})$$

Now: $u \Rightarrow$ Gaussian

$$\langle u(x)\,u(x')\rangle \xrightarrow[|x-x'|>\ell]{} 0$$

$\ell$: correlation decay length

$\boxed{\text{How does } \psi \text{ scale?}}$

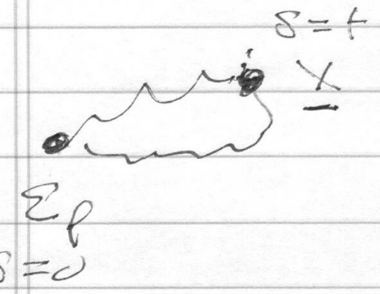
Now, can solve by path integral:

$$\psi(\underline{x},t) = M_{\underline{x}}\left[\exp\left(\int_0^t u(\xi_s)\,ds\right)\psi_0(\xi_s)\right]$$

where

$M_x \equiv$ average over trajectories of Brownian motion:

$$\underline{\xi}_s(x) = (2D)^{1/2}\, W_s(x) \quad \begin{cases} \text{begin at } \xi_\ell \text{ at } s=0 \\ \underline{x} \text{ at } s=t \end{cases}$$



$$\frac{\partial \psi}{\partial t} = D \nabla^2 \psi + U \psi$$

$$= H \psi = (H_0 + U) \psi$$

$$\psi = e^{\int H t}$$

$$= e^{H_0 t} e^{\int U ds}$$

avg over traj. of $H_0$

$$\bar{\psi} = \left\langle e^{\int U ds} \right\rangle$$

Key: Dominant contributions from that trajectory which encounters high value of potential.

$\Rightarrow$ concentration "pt." in traj. space.

Estimate height of maximum, i.e. max U.

Now, for region of radius $R$, s/t $R \gg \ell$, then # cells in $R^3$ ball $\sim (R/\ell)^3$.

(distance)

"Maximum" $\Rightarrow$

$$(R/\ell)^3 P \sim 1$$

Now, $P \sim \exp[-U_0^2/2\sigma^2]$ Gaussian dist $U_0$

so

$$(R/\ell)^3 \exp[-U_0^2/2\sigma^2] \sim 1$$

$$(R/\ell)^3 \sim \exp[U_0^2/2\sigma^2]$$

$$3 \ln(R/\ell) \sim U_0^2/2\sigma^2$$

so

$$\boxed{\max u \sim \left(6\sigma^2 \ln(R/\ell)\right)^{1/2}}$$

Now, $R/\ell \sim (Dt)^{1/2}/\ell$

$$\sim (Dt/\ell^2)^{1/2}$$

so

$$\Psi \sim \exp \int u$$

$$\sim \exp\left[t\left(6\sigma^2 \ln\left(\frac{Dt}{\ell^2}\right)^{1/2}\right)^{1/2}\right]$$

$$\sim \exp\left[t\left(3\sigma^2 \ln(Dt/\ell^2)\right)^{1/2}\right]$$

so

$$\boxed{\Psi \sim \exp\left[t\left(3\sigma^2 \ln(Dt/\ell^2)\right)^{1/2}\right]}$$

$\Rightarrow$ super-exponential growth

(even for typical trajectory)...

And can note that

→ even for $D = 0$, higher moments grow super-exponentially

i.e.,
$$\frac{\partial \psi}{\partial t} = \cancel{D \nabla^2 \psi} + u \psi$$

so
$$\psi = e^{ut} \psi_0$$

⇒
$$\langle \psi^p \rangle = \langle e^{p \overset{\to \text{stoch.}}{u} t} \rangle \psi_0^p$$
$$= \psi_0^p \exp\left[ p^2 \frac{\overline{u^2}}{2} t^2 \right]$$
$$= \psi_0^p \exp\left[ p^2 T^2 t^2 / 2 \right]$$

$$\boxed{\langle \psi^p \rangle^{1/p} \sim \psi_0 \exp\left[ p T^2 t^2 / 2 \right]}$$

→ moments exhibit super-exponential growth

→ grow faster than $\psi$, itself!